A note on the infinitesimal Baker-Campbell-Hausdorff formula

著者	NISHIMURA Hirokazu, TAKAMIYA Hirowaki
page range	1-31
year	2015-07-08
URL	http://hdl.handle.net/2241/00124994

A Note on the Infinitesimal Baker-Campbell-Hausdorff Formula

Hirokazu NISHIMURA[1] and Hirowaki TAKAMIYA[2]
1 Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki 305-8571
Japan
2 Naha Commercial High School, 16-1, Matsuyama-1
Naha, Okinawa 900-0032
Japan

July 9, 2015

Abstract

We have studied the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$ (Math. Appl. **2** (2013), 61-91). In this note we correct some errors in our calculation for $n = 4$ and presents the calculation for $n = 5$ by using Mathematica.

1 Introduction

We are going to work within synthetic differential geometry, in which a *Lie group* G is a group and a microlinear space at the same time. For synthetic differential geometry, the reader is referred to [1] and [3]. Its *Lie algebra* (i.e., its tangent space $\mathbf{T}_e G$ of G at the identity $e \in G$), usually denoted by $\mathfrak{g}$, is endowed with a Lie bracket $[,]$ abiding by antisymmetry and the Jacobi identity. Each element $X \in \mathfrak{g}$ is a mapping $d \in D \mapsto X_d \in G$ with $X_0 = e$, where

$$D = \left\{ d \in \mathbb{R} \mid d^2 = 0 \right\}$$

We assume that the so-called *exponential mapping* $\exp : \mathfrak{g} \to G$ exists. The infinitesimal Baker-Campbell-Hausdorff formula expresses

$$\exp\ (d_1 + ... + d_n)\, X.\exp\ (d_1 + ... + d_n)\, Y$$

as

$$\exp\ (\text{a Lie polynomial of } X \text{ and } Y)$$

where $X, Y \in \mathfrak{g}$ and $d_1, ..., d_n \in D$. In [4] we have calculated the infinitesimal Baker-Campbell-Hausdorff formula up to $n = 4$, but the second author [5] found out some errors in the calculation for $n = 4$.

This paper is based upon [5]. We correct some errors in the calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of $n=4$ in our previous paper [4] and we present a calculation of the infinitesimal Baker-Campbell-Hausdorff formula in case of $n=5$ newly. Both calculations were implemented by using Mathematica.

2 Preliminaries

The infinitesimal Baker-Campbell-Hausdorff formula for $n=3$ goes as follows:

Theorem 1 *(cf. Theorem 7.5 and Theorem 8.3 of [4]) Given $X, Y \in \mathfrak{g}$ and $d_1, d_2, d_3 \in D$, we have*

$$\begin{aligned}&\exp\ (d_1+d_2+d_3)\,X.\exp\ (d_1+d_2+d_3)\,Y\\&=\exp\ (d_1+d_2+d_3)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3)^2\,[X,Y]+\\&\frac{1}{12}\,(d_1+d_2+d_3)^3\,[X-Y,[X,Y]]\end{aligned}$$

The tangent space $\mathbf{T}_X\mathfrak{g}$ of $\mathfrak{g}$ at $X \in \mathfrak{g}$ is naturally identified with $\mathfrak{g}$ itself. That is to say, each $Y \in \mathfrak{g}$ gives rise to $(d \in D \mapsto X + dY \in \mathfrak{g}) \in \mathbf{T}_X\mathfrak{g}$, which yields a bijection between $\mathfrak{g}$ and $\mathbf{T}_X\mathfrak{g}$. Its *left logarithmic derivative* $\delta^{\mathrm{left}}(\exp)$ and its *right logarithmic derivative* $\delta^{\mathrm{right}}(\exp)$ are characterized by the following formulas:

$$\exp\ X+dY=\exp\ X.\left(\delta^{\mathrm{left}}\,(\exp)\,(X)\,(Y)\right)_d \tag{1}$$

and

$$\exp\ X+dY=\left(\delta^{\mathrm{right}}\,(\exp)\,(X)\,(Y)\right)_d.\exp\ X \tag{2}$$

for any $X, Y \in \mathfrak{g}$ and any $d \in D$. For logarithmic derivatives, the reader is referred to §5 of [4] and §38.1 of [2]. We have the following well-known formulas.

Theorem 2 *(cf. Theorem 5.3 and Theorem 5.8 of [4]) Given $X \in \mathfrak{g}$ with $(\mathrm{ad}\,X)^{n+1}$ vanishing for some natural number n, we have*

$$\delta^{\mathrm{left}}\,(\exp)\,(X)=\sum_{p=0}^{n}\frac{(-1)^p}{(p+1)!}\,(\mathrm{ad}\,X)^p$$

and

$$\delta^{\mathrm{right}}\,(\exp)\,(X)=\sum_{p=0}^{n}\frac{1}{(p+1)!}\,(\mathrm{ad}\,X)^p$$

We note in passing that

Proposition 3 *(cf. Proposition 5.4 of [4]) For any $X, Y \in \mathfrak{g}$ with $[X, Y]$ vanishing, we have*

$$\exp\ X.\exp\ Y=\exp\ X+Y$$

The following simple proposition is very useful.

Proposition 4 *(cf. Proposition 4.9 of [4]) For any $X \in \mathfrak{g}$ and any $d \in D$, we have*

$$\exp\ dX = X_d$$

3 The BCH Formula for n=4

Theorem 5

$$\begin{aligned}
&\exp\ (d_1 + d_2 + d_3 + d_4)\, X . \exp\ (d_1 + d_2 + d_3 + d_4)\, Y \\
&= \exp\ (d_1 + d_2 + d_3 + d_4)\,(X + Y) \\
&+ \frac{1}{2}\,(d_1 + d_2 + d_3 + d_4)^2\,[X, Y] \\
&+ \frac{1}{12}\,(d_1 + d_2 + d_3 + d_4)^3\,[[X, Y]\,, Y - X] \\
&+ \frac{1}{96}\,(d_1 + d_2 + d_3 + d_4)^4\,([X + Y, [[X, Y]\,, X + Y]] + [[[X, Y]\,, X - Y]\,, X - Y])
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\exp\ (d_1 + d_2 + d_3 + d_4)\, X . \exp\ (d_1 + d_2 + d_3 + d_4)\, Y \\
&= \exp\ (d_1 + d_2 + d_3)\, X + d_4 X . \exp\ (d_1 + d_2 + d_3)\, Y + d_4 Y \\
&= \exp\ d_4 X . \exp\ (d_1 + d_2 + d_3)\, X . \exp\ (d_1 + d_2 + d_3)\, Y . \exp\ d_4 Y \\
&)\text{By Proposition 3}(\\
&= \exp\ d_4 X . \\
&\exp\ \left((d_1 + d_2 + d_3)\,(X + Y) + \frac{1}{2}\,(d_1 + d_2 + d_3)^2\,[X, Y] + \frac{1}{2} d_1 d_2 d_3\,[[X, Y]\,, Y - X] \right) . \\
&\exp\ d_4 Y \\
&)\text{By Theorem 1}(\qquad (3)
\end{aligned}$$

By the way, due to Theorem 2, we have

$$\delta^{\text{left}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]\end{array}\right)(Y)$$
$$=-\frac{1}{2}d_1[X,Y]-\frac{1}{2}d_2[X,Y]+\frac{1}{3}d_1d_2[X,[X,Y]]+\frac{1}{3}d_1d_2[Y,[X,Y]]$$
$$-\frac{1}{2}d_1d_2[[X,Y],Y]-\frac{1}{2}d_3[X,Y]+\frac{1}{3}d_1d_3[X,[X,Y]]+\frac{1}{3}d_1d_3[Y,[X,Y]]$$
$$-\frac{1}{2}d_1d_3[[X,Y],Y]+\frac{1}{3}d_2d_3[X,[X,Y]]+\frac{1}{3}d_2d_3[Y,[X,Y]]$$
$$-\frac{1}{2}d_2d_3[[X,Y],Y]-\frac{1}{4}d_1d_2d_3[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[X,[Y,[X,Y]]]$$
$$+\frac{1}{2}d_1d_2d_3[X,[[X,Y],Y]]-\frac{1}{4}d_1d_2d_3[Y,[X,[X,Y]]]$$
$$-\frac{1}{4}d_1d_2d_3[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[Y,[[X,Y],Y]]$$
$$+\frac{1}{4}d_1d_2d_3[[[X,Y],X],Y]-\frac{1}{4}d_1d_2d_3[[[X,Y],Y],Y]+Y \tag{4}$$

Letting n_{41} be the right-hand side of (4) with the last term Y deleted, we have

(3)
$= \exp\ d_4X.$
$\exp\left((d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]\right).$
$\exp\ d_4(n_{41}+Y).\exp\ -d_4n_{41}$
)By Proposition 3(
$= \exp\ d_4X.$
$\exp\left((d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]\right).$
$(n_{41}+Y)_{d_4}.\exp\ -d_4n_{41}$
)By Proposition 4(
$$=\exp\ d_4X.\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y\end{array}\right).\exp\ -d_4n_{41}$$
)By (4)((5)

We let i_{41} be the result of n_{41} by deleting all the terms whose coefficients contain $d_1d_2d_3$. Then, due to Theorem 2, we have

$$\begin{aligned}
&\delta^{\mathrm{left}}(\exp)\begin{pmatrix} (d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y] \\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y \end{pmatrix}(i_{41}) \\
&= \frac{1}{2}d_1[X,Y]+\frac{1}{2}d_2[X,Y]-\frac{5}{6}d_1d_2[X,[X,Y]]-\frac{5}{6}d_1d_2[Y,[X,Y]] \\
&+\frac{1}{2}d_1d_2[[X,Y],Y]+\frac{1}{2}d_3[X,Y]-\frac{5}{6}d_1d_3[X,[X,Y]]-\frac{5}{6}d_1d_3[Y,[X,Y]] \\
&+\frac{1}{2}d_1d_3[[X,Y],Y]-\frac{5}{6}d_2d_3[X,[X,Y]]-\frac{5}{6}d_2d_3[Y,[X,Y]] \\
&+\frac{1}{2}d_2d_3[[X,Y],Y]+d_1d_2d_3[X,[X,[X,Y]]] \\
&+d_1d_2d_3[X,[Y,[X,Y]]]-\frac{3}{4}d_1d_2d_3[X,[[X,Y],Y]]+d_1d_2d_3[Y,[X,[X,Y]]] \\
&+d_1d_2d_3[Y,[Y,[X,Y]]]-\frac{3}{4}d_1d_2d_3[Y,[[X,Y],Y]] \qquad (6)
\end{aligned}$$

Letting n_{42} be the right-hand side of (6), we have

$$\begin{aligned}
&(5) \\
&= \exp\ d_4X.\exp\begin{pmatrix} (d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y] \\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y \end{pmatrix}. \\
&\exp\ d_4n_{42}.\exp\ -d_4n_{42}-d_4n_{41} \\
&)\text{By Proposition 3}(\\
&= \exp\ d_4X.\exp\begin{pmatrix} (d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y] \\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y \end{pmatrix}. \\
&(n_{42})_{d_4}.\exp\ -d_4n_{42}-d_4n_{41} \\
&)\text{By Proposition 4}(\\
&= \exp\ d_4X.\exp\begin{pmatrix} (d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y] \\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41} \end{pmatrix}. \\
&\exp\ -d_4n_{42}-d_4n_{41} \\
&)\text{By (6)}(\qquad (7)
\end{aligned}$$

We let i_{42} be the result of $-n_{42}-n_{41}$ by deleting all the terms whose coefficients contain $d_1d_2d_3$. Then, thanks to Theorem 2, we have

$$\delta^{\mathrm{left}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}\end{array}\right)(i_{42})$$
$$=\frac{1}{2}d_1d_2[X,[X,Y]]+\frac{1}{2}d_1d_2[Y,[X,Y]]+\frac{1}{2}d_1d_3[X,[X,Y]]$$
$$+\frac{1}{2}d_1d_3[Y,[X,Y]]+\frac{1}{2}d_2d_3[X,[X,Y]]+\frac{1}{2}d_2d_3[Y,[X,Y]]$$
$$-\frac{3}{4}d_1d_2d_3[X,[X,[X,Y]]]-\frac{3}{4}d_1d_2d_3[X,[Y,[X,Y]]]$$
$$-\frac{3}{4}d_1d_2d_3[Y,[X,[X,Y]]]-\frac{3}{4}d_1d_2d_3[Y,[Y,[X,Y]]] \tag{8}$$

Letting n_{43} be the right-hand side of (8), we have

(7)

$$=\exp\ d_4X.\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}\end{array}\right).$$

$\exp\ d_4n_{43}.\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$

)By Proposition 3(

$$=\exp\ d_4X.\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}\end{array}\right).$$

$(n_{43})_{d_4}.\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$

)By Proposition 4(

$$=\exp\ d_4X.\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\+\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}+d_4i_{42}\end{array}\right).$$

$\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$

)By (8)((9)

Since the coefficient of every term in $-n_{43}-n_{42}-n_{41}$ contains $d_1d_2d_3$, we now turn our attention to the left $\exp\ d_4X$. Now, thanks to Theorem 2, we have

$$
\begin{aligned}
&\delta^{\text{right}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}+d_4i_{42}\end{array}\right)(X)\\
&=-\frac{1}{2}d_1[X,Y]-\frac{1}{2}d_2[X,Y]-\frac{1}{3}d_1d_2[X,[X,Y]]-\frac{1}{3}d_1d_2[Y,[X,Y]]\\
&+\frac{1}{2}d_1d_2[[X,Y],X]-\frac{1}{2}d_3[X,Y]-\frac{1}{3}d_1d_3[X,[X,Y]]-\frac{1}{3}d_1d_3[Y,[X,Y]]\\
&+\frac{1}{2}d_1d_3[[X,Y],X]-\frac{1}{3}d_2d_3[X,[X,Y]]-\frac{1}{3}d_2d_3[Y,[X,Y]]\\
&+\frac{1}{2}d_2d_3[[X,Y],X]-\frac{1}{4}d_1d_2d_3[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[X,[Y,[X,Y]]]\\
&+\frac{1}{2}d_1d_2d_3[X,[[X,Y],X]]-\frac{1}{4}d_1d_2d_3[Y,[X,[X,Y]]]\\
&-\frac{1}{4}d_1d_2d_3[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[Y,[[X,Y],X]]\\
&-\frac{1}{2}d_1d_2d_3[[X,Y],[X,Y]]-\frac{1}{4}d_1d_2d_3[[[X,Y],X],X]\\
&+\frac{1}{4}d_1d_2d_3[[[X,Y],Y],X]+X
\end{aligned}
\tag{10}
$$

We let m_{41} be the right-hand side of (10) with the last term X deleted. Then we have

$$
\begin{aligned}
&(9)\\
&=\exp\ -d_4m_{41}.\exp\ d_4X+d_4m_{41}.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}+d_4i_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By Proposition 3}(\\
&=\exp\ -d_4m_{41}.(X+m_{41})_{d_4}.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4Y+d_4i_{41}+d_4i_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By Proposition 4}(\\
&=\exp\ -d_4m_{41}.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+\frac{1}{2}(d_1+d_2+d_3)^2[X,Y]\\ +\frac{1}{2}d_1d_2d_3[[X,Y],Y-X]+d_4i_{41}+d_4i_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By (10)}(
\end{aligned}
\tag{11}
$$

We let j_{41} be the result of $-m_{41}$ by deleting all the terms whose coefficients contain $d_1d_2d_3$. Then, thanks to Theorem 2, we have

$$\delta^{\mathrm{right}}\left(\exp\right)\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4i_{42}\end{array}\right)\left(j_{41}\right)$$
$$=\frac{1}{2}d_1\left[X,Y\right]+\frac{1}{2}d_2\left[X,Y\right]+\frac{5}{6}d_1d_2\left[X,\left[X,Y\right]\right]+\frac{5}{6}d_1d_2\left[Y,\left[X,Y\right]\right]$$
$$-\frac{1}{2}d_1d_2\left[\left[X,Y\right],X\right]+\frac{1}{2}d_3\left[X,Y\right]+\frac{5}{6}d_1d_3\left[X,\left[X,Y\right]\right]$$
$$+\frac{5}{6}d_1d_3\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_1d_3\left[\left[X,Y\right],X\right]+\frac{5}{6}d_2d_3\left[X,\left[X,Y\right]\right]$$
$$+\frac{5}{6}d_2d_3\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_2d_3\left[\left[X,Y\right],X\right]+d_1d_2d_3\left[X,\left[X,\left[X,Y\right]\right]\right]$$
$$+d_1d_2d_3\left[X,\left[Y,\left[X,Y\right]\right]\right]-\frac{3}{4}d_1d_2d_3\left[X,\left[\left[X,Y\right],X\right]\right]$$
$$+d_1d_2d_3\left[Y,\left[X,\left[X,Y\right]\right]\right]+d_1d_2d_3\left[Y,\left[Y,\left[X,Y\right]\right]\right]$$
$$-\frac{3}{4}d_1d_2d_3\left[Y,\left[\left[X,Y\right],X\right]\right]+\frac{3}{4}d_1d_2d_3\left[\left[X,Y\right],\left[X,Y\right]\right] \qquad (12)$$

Letting m_{42} be the right-hand side of (12), we have

(11)
$$=\exp\ -d_4m_{41}-d_4m_{42}.\exp\ d_4m_{42}.$$
$$\exp\ \left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4i_{42}\end{array}\right).$$
$$\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$$
)By Proposition 3(
$$=\exp\ -d_4m_{41}-d_4m_{42}.\left(m_{42}\right)_{d_4}.$$
$$\exp\ \left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4i_{42}\end{array}\right).$$
$$\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$$
)By Proposition 4(
$$=\exp\ -d_4m_{41}-d_4m_{42}.$$
$$\exp\ \left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}\end{array}\right).$$
$$\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}$$
)By (12)((13)

We let j_{42} be the result of $-m_{41}-m_{42}$ by deleting all the terms whose coefficients contain $d_1 d_2 d_3$. Then, thanks to Theorem 2, we have

$$
\begin{aligned}
&\delta^{\mathrm{right}}\left(\exp\right)\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}\end{array}\right)\left(j_{42}\right)\\
&=-\frac{1}{2}d_1d_2\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_1d_2\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_1d_3\left[X,\left[X,Y\right]\right]\\
&-\frac{1}{2}d_1d_3\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_2d_3\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_2d_3\left[Y,\left[X,Y\right]\right]\\
&-\frac{3}{4}d_1d_2d_3\left[X,\left[X,\left[X,Y\right]\right]\right]-\frac{3}{4}d_1d_2d_3\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&-\frac{3}{4}d_1d_2d_3\left[Y,\left[X,\left[X,Y\right]\right]\right]-\frac{3}{4}d_1d_2d_3\left[Y,\left[Y,\left[X,Y\right]\right]\right]
\end{aligned}
\tag{14}
$$

Letting m_{43} be the right-hand side of (14), we have

$$
\begin{aligned}
&(13)\\
&=\exp\ -d_4m_{41}-d_4m_{42}-d_4m_{43}.\exp\ d_4m_{43}.\\
&\exp\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By Proposition 3}(\\
&=\exp\ -d_4m_{41}-d_4m_{42}-d_4m_{43}.\left(m_{43}\right)_{d_4}\\
&\exp\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By Proposition 4}(\\
&=\exp\ -d_4m_{41}-d_4m_{42}-d_4m_{43}.\\
&\exp\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}+d_4j_{42}\end{array}\right).\\
&\exp\ -d_4n_{43}-d_4n_{42}-d_4n_{41}\\
&)\text{By (14)}(
\end{aligned}
\tag{15}
$$

Since the coefficient of every term in $-m_{41}-m_{42}-m_{43}$ contains $d_1d_2d_3$, we are done, so that we have

$$
\begin{aligned}
&(15)\\
&=\exp\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left[X,Y\right]\\+\frac{1}{2}d_1d_2d_3\left[\left[X,Y\right],Y-X\right]+d_4i_{41}+d_4j_{41}+d_4i_{42}+d_4j_{42}\\-d_4m_{41}-d_4m_{42}-d_4m_{43}-d_4n_{43}-d_4n_{42}-d_4n_{41}\end{array}\right)
\end{aligned}
\tag{16}
$$

It is easy to see that

$$
\begin{aligned}
& j_{41}+i_{42}+j_{42} \\
& =d_1[X,Y]+d_2[X,Y]-\frac{1}{2}d_1d_2[[X,Y],X]+\frac{1}{2}d_1d_2[[X,Y],Y] \\
& d_3[X,Y]-\frac{1}{2}d_1d_3[[X,Y],X]+\frac{1}{2}d_1d_3[[X,Y],Y] \\
& -\frac{1}{2}d_2d_3[[X,Y],X]+\frac{1}{2}d_2d_3[[X,Y],Y]
\end{aligned}
$$

whereas

$$
\begin{aligned}
& -m_{41}-m_{42}-m_{43}-n_{43}-n_{42}-n_{41} \\
& =\frac{1}{4}d_1d_2d_3[X,[[X,Y],X]]+\frac{1}{4}d_1d_2d_3[X,[[X,Y],Y]] \\
& +\frac{1}{4}d_1d_2d_3[Y,[[X,Y],X]]+\frac{1}{4}d_1d_2d_3[Y,[[X,Y],Y]] \\
& +\frac{1}{4}d_1d_2d_3[[[X,Y],X],X]-\frac{1}{4}d_1d_2d_3[[[X,Y],X],Y] \\
& -\frac{1}{4}d_1d_2d_3[[[X,Y],Y],X]+\frac{1}{4}d_1d_2d_3[[[X,Y],Y],Y]
\end{aligned}
$$

Therefore we have the desired result. ■

4 The BCH Formula for n=5

Theorem 6

$$
\begin{aligned}
& \exp\ (d_1+d_2+d_3+d_4+d_5)X.\exp\ (d_1+d_2+d_3+d_4+d_5)Y \\
& =\exp\ (d_1+d_2+d_3+d_4+d_5)(X+Y) \\
& +\frac{1}{2}(d_1+d_2+d_3+d_4+d_5)^2[X,Y] \\
& +\frac{1}{12}(d_1+d_2+d_3+d_4+d_5)^3[[X,Y],Y-X] \\
& +\frac{1}{96}(d_1+d_2+d_3+d_4+d_5)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\
& +\frac{1}{120}(d_1+d_2+d_3+d_4+d_5)^5\begin{pmatrix} \frac{5}{6}[X+Y,[X+Y,[[X,Y],X-Y]]] \\ +\frac{1}{2}[[X,Y],[[X,Y],X+Y]] \\ +\frac{1}{8}[[X+Y,[[X,Y],X+Y]],Y-X] \\ +\frac{1}{8}[[[[X,Y],Y-X],X-Y],X-Y] \end{pmatrix}
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\exp\ (d_1+d_2+d_3+d_4+d_5)\,X.\exp\ (d_1+d_2+d_3+d_4+d_5)\,Y \\
&=\exp\ (d_1+d_2+d_3+d_4)\,X+d_5X.\exp\ (d_1+d_2+d_3+d_4)\,Y+d_5Y \\
&=\exp\ d_5X.\exp\ (d_1+d_2+d_3+d_4)\,X.\exp\ (d_1+d_2+d_3+d_4)\,Y.\exp\ d_5Y \\
&\text{)By Proposition 3(} \\
&=\exp\ d_5X. \\
&\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y] \\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y]\,,Y-X] \\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y]\,,X+Y]]+[[[X,Y]\,,X-Y]\,,X-Y]) \end{array}\right). \\
&\exp\ d_5Y \\
&\text{)By Theorem 5(}
\end{aligned} \tag{17}
$$

By the way, due to Theorem 2, we have

$$\delta^{\mathrm{left}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[X+Y,[[X,Y],X+Y]]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[[[X,Y],X-Y],X-Y]\end{array}\right)(Y)$$

$$= Y-\frac{1}{2}d_1[X,Y]-\frac{1}{2}d_2[X,Y]+\frac{1}{3}d_1d_2[X,[X,Y]]+\frac{1}{3}d_1d_2[Y,[X,Y]]$$
$$-\frac{1}{2}d_1d_2[[X,Y],Y]-\frac{1}{2}d_3[X,Y]+\frac{1}{3}d_1d_3[X,[X,Y]]+\frac{1}{3}d_1d_3[Y,[X,Y]]$$
$$-\frac{1}{2}d_1d_3[[X,Y],Y]+\frac{1}{3}d_2d_3[X,[X,Y]]+\frac{1}{3}d_2d_3[Y,[X,Y]]-\frac{1}{2}d_2d_3[[X,Y],Y]$$
$$-\frac{1}{4}d_1d_2d_3[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[X,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[X,[[X,Y],Y]]$$
$$-\frac{1}{4}d_1d_2d_3[Y,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[Y,[[X,Y],Y]]$$
$$+\frac{1}{4}d_1d_2d_3[[[X,Y],X],Y]-\frac{1}{4}d_1d_2d_3[[[X,Y],Y],Y]-\frac{1}{2}d_4[X,Y]+\frac{1}{3}d_1d_4[X,[X,Y]]$$
$$+\frac{1}{3}d_1d_4[Y,[X,Y]]-\frac{1}{2}d_1d_4[[X,Y],Y]+\frac{1}{3}d_2d_4[X,[X,Y]]+\frac{1}{3}d_2d_4[Y,[X,Y]]$$
$$-\frac{1}{2}d_2d_4[[X,Y],Y]-\frac{1}{4}d_1d_2d_4[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_4[X,[Y,[X,Y]]]$$
$$+\frac{1}{2}d_1d_2d_4[X,[[X,Y],Y]]-\frac{1}{4}d_1d_2d_4[Y,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_4[Y,[Y,[X,Y]]]$$
$$+\frac{1}{2}d_1d_2d_4[Y,[[X,Y],Y]]+\frac{1}{4}d_1d_2d_4[[[X,Y],X],Y]-\frac{1}{4}d_1d_2d_4[[[X,Y],Y],Y]$$
$$+\frac{1}{3}d_3d_4[X,[X,Y]]+\frac{1}{3}d_3d_4[Y,[X,Y]]-\frac{1}{2}d_3d_4[[X,Y],Y]-\frac{1}{4}d_1d_3d_4[X,[X,[X,Y]]]$$
$$-\frac{1}{4}d_1d_3d_4[X,[Y,[X,Y]]]+\frac{1}{2}d_1d_3d_4[X,[[X,Y],Y]]-\frac{1}{4}d_1d_3d_4[Y,[X,[X,Y]]]$$
$$-\frac{1}{4}d_1d_3d_4[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_3d_4[Y,[[X,Y],Y]]+\frac{1}{4}d_1d_3d_4[[[X,Y],X],Y]$$
$$-\frac{1}{4}d_1d_3d_4[[[X,Y],Y],Y]-\frac{1}{4}d_2d_3d_4[X,[X,[X,Y]]]-\frac{1}{4}d_2d_3d_4[X,[Y,[X,Y]]]$$
$$+\frac{1}{2}d_2d_3d_4[X,[[X,Y],Y]]-\frac{1}{4}d_2d_3d_4[Y,[X,[X,Y]]]-\frac{1}{4}d_2d_3d_4[Y,[Y,[X,Y]]]$$
$$+\frac{1}{2}d_2d_3d_4[Y,[[X,Y],Y]]+\frac{1}{4}d_2d_3d_4[[[X,Y],X],Y]-\frac{1}{4}d_2d_3d_4[[[X,Y],Y],Y]$$
$$+\frac{1}{5}d_1d_2d_3d_4[X,[X,[X,[X,Y]]]]+\frac{1}{5}d_1d_2d_3d_4[X,[X,[Y,[X,Y]]]]$$
$$-\frac{1}{2}d_1d_2d_3d_4[X,[X,[[X,Y],Y]]]+\frac{1}{5}d_1d_2d_3d_4[X,[Y,[X,[X,Y]]]]$$
$$+\frac{1}{5}d_1d_2d_3d_4[X,[Y,[Y,[X,Y]]]]-\frac{1}{2}d_1d_2d_3d_4[X,[Y,[[X,Y],Y]]]$$
$$-\frac{1}{3}d_1d_2d_3d_4[X,[[[X,Y],X],Y]]+\frac{1}{3}d_1d_2d_3d_4[X,[[[X,Y],Y],Y]]$$

$$
\begin{aligned}
&+\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]+\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],Y\right]\right]\right]+\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right]\\
&-\frac{1}{3}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],X\right],Y\right]\right]+\frac{1}{3}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],Y\right],Y\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right]\\
&+d_1d_2d_3d_4\left[\left[X,Y\right],\left[\left[X,Y\right],Y\right]\right]-\frac{1}{8}d_1d_2d_3d_4\left[\left[X,\left[\left[X,Y\right],X\right]\right],Y\right]\\
&-\frac{1}{8}d_1d_2d_3d_4\left[\left[X,\left[\left[X,Y\right],Y\right]\right],Y\right]-\frac{1}{8}d_1d_2d_3d_4\left[\left[Y,\left[\left[X,Y\right],X\right]\right],Y\right]\\
&-\frac{1}{8}d_1d_2d_3d_4\left[\left[Y,\left[\left[X,Y\right],Y\right]\right],Y\right]-\frac{1}{3}d_1d_2d_3d_4\left[\left[\left[X,Y\right],X\right],\left[X,Y\right]\right]\\
&+\frac{1}{3}d_1d_2d_3d_4\left[\left[\left[X,Y\right],Y\right],\left[X,Y\right]\right]-\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],X\right],X\right],Y\right]\\
&+\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],X\right],Y\right],Y\right]+\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],Y\right],X\right],Y\right]\\
&-\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],Y\right],Y\right],Y\right] \qquad (18)
\end{aligned}
$$

Letting n_{51} be the right-hand side of (18) with the first term Y deleted, we have

$$
\begin{aligned}
&(17)\\
&= \exp\ d_5X.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y]\\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y]\,,Y-X]\\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y]\,,X+Y]]+[[[X,Y]\,,X-Y]\,,X-Y])\end{array}\right).\\
&\exp\ d_5X+d_5n_{51}.\exp\ -d_5n_{51}\\
&)\text{By Proposition 3}(\\
&= \exp\ d_5X.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)\,(X+Y)+d_5Y+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y]\\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y]\,,Y-X]\\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y]\,,X+Y]]+[[[X,Y]\,,X-Y]\,,X-Y])\end{array}\right).\\
&\exp\ -d_5n_{51}\\
&)\text{By (18)}(\qquad (19)
\end{aligned}
$$

We let i_{51} be the result of $-n_{51}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, by dint of Theorem 2, we have

$$\delta^{\mathrm{left}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[X+Y,[[X,Y],X+Y]]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[[[X,Y],X-Y],X-Y]\end{array}\right)(i_{51})$$

$$\begin{aligned}
&=\frac{1}{2}d_1[X,Y]+\frac{1}{2}d_2[X,Y]-\frac{5}{6}d_1d_2[X,[X,Y]]-\frac{5}{6}d_1d_2[Y,[X,Y]]+\frac{1}{2}d_1d_2[[X,Y],Y]\\
&+\frac{1}{2}d_3[X,Y]-\frac{5}{6}d_1d_3[X,[X,Y]]-\frac{5}{6}d_1d_3[Y,[X,Y]]+\frac{1}{2}d_1d_3[[X,Y],Y]\\
&-\frac{5}{6}d_2d_3[X,[X,Y]]-\frac{5}{6}d_2d_3[Y,[X,Y]]+\frac{1}{2}d_2d_3[[X,Y],Y]+\frac{5}{4}d_1d_2d_3[X,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_2d_3[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_3[X,[[X,Y],Y]]+\frac{5}{4}d_1d_2d_3[Y,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_2d_3[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_3[Y,[[X,Y],Y]]-\frac{1}{4}d_1d_2d_3[[[X,Y],X],Y]\\
&+\frac{1}{4}d_1d_2d_3[[[X,Y],Y],Y]+\frac{1}{2}d_4[X,Y]-\frac{5}{6}d_1d_4[X,[X,Y]]-\frac{5}{6}d_1d_4[Y,[X,Y]]\\
&+\frac{1}{2}d_1d_4[[X,Y],Y]-\frac{5}{6}d_2d_4[X,[X,Y]]-\frac{5}{6}d_2d_4[Y,[X,Y]]+\frac{1}{2}d_2d_4[[X,Y],Y]\\
&+\frac{5}{4}d_1d_2d_4[X,[X,[X,Y]]]+\frac{5}{4}d_1d_2d_4[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_4[X,[[X,Y],Y]]\\
&+\frac{5}{4}d_1d_2d_4[Y,[X,[X,Y]]]+\frac{5}{4}d_1d_2d_4[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_4[Y,[[X,Y],Y]]\\
&-\frac{1}{4}d_1d_2d_4[[[X,Y],X],Y]+\frac{1}{4}d_1d_2d_4[[[X,Y],Y],Y]-\frac{5}{6}d_3d_4[X,[X,Y]]\\
&-\frac{5}{6}d_3d_4[Y,[X,Y]]+\frac{1}{2}d_3d_4[[X,Y],X]+\frac{5}{4}d_1d_3d_4[X,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_3d_4[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_3d_4[X,[[X,Y],Y]]+\frac{5}{4}d_1d_3d_4[Y,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_3d_4[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_3d_4[Y,[[X,Y],Y]]-\frac{1}{4}d_1d_3d_4[[[X,Y],X],Y]\\
&+\frac{1}{4}d_1d_3d_4[[[X,Y],Y],Y]+\frac{5}{4}d_2d_3d_4[X,[X,[X,Y]]]+\frac{5}{4}d_2d_3d_4[X,[Y,[X,Y]]]\\
&-\frac{5}{4}d_2d_3d_4[X,[[X,Y],Y]]+\frac{5}{4}d_2d_3d_4[Y,[X,[X,Y]]]+\frac{5}{4}d_2d_3d_4[Y,[Y,[X,Y]]]\\
&-\frac{5}{4}d_2d_3d_4[Y,[[X,Y],Y]]-\frac{1}{4}d_2d_3d_4[[[X,Y],X],Y]+\frac{1}{4}d_2d_3d_4[[[X,Y],Y],Y]\\
&-\frac{5}{3}d_1d_2d_3d_4[X,[X,[X,[X,Y]]]]-\frac{5}{3}d_1d_2d_3d_4[X,[X,[Y,[X,Y]]]]\\
&+2d_1d_2d_3d_4[X,[X,[[X,Y],Y]]]-\frac{5}{3}d_1d_2d_3d_4[X,[Y,[X,[X,Y]]]]\\
&-\frac{5}{3}d_1d_2d_3d_4[X,[Y,[Y,[X,Y]]]]+2d_1d_2d_3d_4[X,[Y,[[X,Y],Y]]]\\
&+\frac{1}{2}d_1d_2d_3d_4[X,[[[X,Y],X],Y]]-\frac{1}{2}d_1d_2d_3d_4[X,[[[X,Y],Y],Y]]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]-\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&+2d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],Y\right]\right]\right]-\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&-\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]+2d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],X\right],Y\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],Y\right],Y\right]\right]\\
&+2d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]+2d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right]\\
&-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[\left[X,Y\right],Y\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[\left[\left[X,Y\right],X\right],\left[X,Y\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[\left[\left[X,Y\right],Y\right],\left[X,Y\right]\right] \qquad (20)
\end{aligned}
$$

Letting n_{52} be the right-hand side of (20), we have

$$
\begin{aligned}
&(19)\\
&=\exp\ d_5X.\\
&\exp\left(\begin{array}{c}
(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\
+\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\
+\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])
\end{array}\right).\\
&\exp\ d_5n_{52}.\exp\ -d_5n_{51}-d_5n_{52}\\
&)\text{By Proposition 3}(\\
&=\exp\ d_5X.\\
&\exp\left(\begin{array}{c}
(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\
+\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\
+\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])
\end{array}\right).\\
&(n_{52})_{d_5}.\exp\ -d_5n_{51}-d_5n_{52}\\
&)\text{By Proposition 4}(\\
&=\exp\ d_5X.\\
&\exp\left(\begin{array}{c}
(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\
+\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\
+\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\
+d_5i_{51}
\end{array}\right).\\
&\exp\ -d_5n_{51}-d_5n_{52}\\
&)\text{By (20)}(\qquad (21)
\end{aligned}
$$

We let i_{52} be the result of $-n_{51}-n_{52}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, thanks to Theorem 2, we have

$$\delta^{\mathrm{left}}\left(\exp\right)\left(\begin{array}{c} \left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+d_5Y+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right] \\ +\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right] \\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right] \\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]+d_5i_{51} \end{array}\right)\left(i_{52}\right)$$

$$\begin{aligned}
&=\frac{1}{2}d_1d_2\left[X,\left[X,Y\right]\right]+\frac{1}{2}d_1d_2\left[Y,\left[X,Y\right]\right]+\frac{1}{2}d_1d_3\left[X,\left[X,Y\right]\right]+\frac{1}{2}d_1d_3\left[Y,\left[X,Y\right]\right]\\
&+\frac{1}{2}d_2d_3\left[X,\left[X,Y\right]\right]+\frac{1}{2}d_2d_3\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_2d_3\left[X,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_3\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[X,\left[\left[X,Y\right],Y\right]\right]-\frac{7}{4}d_1d_2d_3\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_3\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[Y,\left[\left[X,Y\right],Y\right]\right]+\frac{1}{2}d_1d_4\left[X,\left[X,Y\right]\right]\\
&+\frac{1}{2}d_1d_4\left[Y,\left[X,Y\right]\right]+\frac{1}{2}d_2d_4\left[X,\left[X,Y\right]\right]+\frac{1}{2}d_2d_4\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_2d_4\left[X,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[X,\left[\left[X,Y\right],Y\right]\right]-\frac{7}{4}d_1d_2d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[Y,\left[\left[X,Y\right],Y\right]\right]+\frac{1}{2}d_3d_4\left[X,\left[X,Y\right]\right]\\
&+\frac{1}{2}d_3d_4\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_1d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_3d_4\left[X,\left[\left[X,Y\right],Y\right]\right]-\frac{7}{4}d_1d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_1d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_3d_4\left[Y,\left[\left[X,Y\right],Y\right]\right]-\frac{7}{4}d_2d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_2d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[X,\left[\left[X,Y\right],Y\right]\right]-\frac{7}{4}d_2d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_2d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[Y,\left[\left[X,Y\right],Y\right]\right]+3d_1d_2d_3d_4\left[X,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+3d_1d_2d_3d_4\left[X,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[X,\left[\left[X,Y\right],Y\right]\right]\right]\\
&+3d_1d_2d_3d_4\left[X,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]+3d_1d_2d_3d_4\left[X,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[X,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right]+3d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+3d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],Y\right]\right]\right]\\
&+3d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]+3d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right]-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right] \qquad (22)
\end{aligned}$$

Letting n_{53} be the right-hand side of (22), we have

(21)

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c} \left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+d_5Y+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right] \\ +\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right] \\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left(\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right]+\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]\right) \\ +d_5i_{51} \end{array}\right).$$

$\exp\ d_5n_{53}. \exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}$

)By Proposition 3(

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c} \left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+d_5Y+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right] \\ +\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right] \\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left(\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right]+\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]\right) \\ +d_5i_{51} \end{array}\right).$$

$\left(n_{53}\right)_{d_5}. \exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}$

)By Proposition 4(

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c} \left(d_1+d_2+d_3+d_4\right)\left(X+Y\right)+d_5Y+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right] \\ +\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right] \\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left(\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right]+\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]\right) \\ +d_5i_{51}+d_5i_{52} \end{array}\right).$$

$\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}$

)By (22)((23)

We let i_{53} be the result of $-n_{51}-n_{52}-n_{53}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, due to Theorem 2, we have

$$\delta^{\text{left}}(\exp)\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[X+Y,[[X,Y],X+Y]]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[[[X,Y],X-Y],X-Y]\\ +d_5i_{51}+d_5i_{52}\end{array}\right)(i_{53})$$

$$\begin{aligned}
&=\frac{3}{4}d_1d_2d_3[X,[X,[X,Y]]]+\frac{3}{4}d_1d_2d_3[X,[Y,[X,Y]]]+\frac{3}{4}d_1d_2d_3[Y,[X,[X,Y]]]\\
&+\frac{3}{4}d_1d_2d_3[Y,[Y,[X,Y]]]+\frac{3}{4}d_1d_2d_4[X,[X,[X,Y]]]+\frac{3}{4}d_1d_2d_4[X,[Y,[X,Y]]]\\
&+\frac{3}{4}d_1d_2d_4[Y,[X,[X,Y]]]+\frac{3}{4}d_1d_2d_4[Y,[Y,[X,Y]]]+\frac{3}{4}d_1d_3d_4[X,[X,[X,Y]]]\\
&+\frac{3}{4}d_1d_3d_4[X,[Y,[X,Y]]]+\frac{3}{4}d_1d_3d_4[Y,[X,[X,Y]]]+\frac{3}{4}d_1d_3d_4[Y,[Y,[X,Y]]]\\
&+\frac{3}{4}d_2d_3d_4[X,[X,[X,Y]]]+\frac{3}{4}d_2d_3d_4[X,[Y,[X,Y]]]+\frac{3}{4}d_2d_3d_4[Y,[X,[X,Y]]]\\
&+\frac{3}{4}d_2d_3d_4[Y,[Y,[X,Y]]]-\frac{1}{2}d_1d_2d_3d_4[X,[X,[X,[X,Y]]]]\\
&-\frac{1}{2}d_1d_2d_3d_4[X,[X,[Y,[X,Y]]]]-\frac{1}{2}d_1d_2d_3d_4[X,[Y,[X,[X,Y]]]]\\
&-\frac{1}{2}d_1d_2d_3d_4[X,[Y,[Y,[X,Y]]]]-\frac{1}{2}d_1d_2d_3d_4[Y,[X,[X,[X,Y]]]]\\
&-\frac{1}{2}d_1d_2d_3d_4[Y,[X,[Y,[X,Y]]]]-\frac{1}{2}d_1d_2d_3d_4[Y,[Y,[X,[X,Y]]]]\\
&-\frac{1}{2}d_1d_2d_3d_4[Y,[Y,[Y,[X,Y]]]] \qquad (24)
\end{aligned}$$

Letting n_{54} be the right-hand side of (24), we have

(23)

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}\end{array}\right).$$

$\exp\ d_5n_{54}.\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}$

)By Proposition 3(

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}\end{array}\right).$$

$(n_{54})_{d_5}.\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}$

)By Proposition 4(

$= \exp\ d_5X.$

$$\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y]\\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X]\\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}+d_5i_{53}\end{array}\right).$$

$\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}$

)By (24)((25)

Since the coefficient of every term in $-n_{51}-n_{52}-n_{53}-n_{54}$ contains $d_1d_2d_3d_4$, we turn our attention to the left exp d_5X. Now, thanks to Theorem 2, we have

$$\delta^{\mathrm{right}}(\exp)\begin{pmatrix} (d_1+d_2+d_3+d_4)(X+Y)+d_5Y+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[X+Y,[[X,Y],X+Y]] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[[[X,Y],X-Y],X-Y] \\ +d_5i_{51}+d_5i_{52}+d_5i_{53} \end{pmatrix}(X)$$

$$\begin{aligned}
&= X-\frac{1}{2}d_1[X,Y]-\frac{1}{2}d_2[X,Y]-\frac{1}{3}d_1d_2[X,[X,Y]]-\frac{1}{3}d_1d_2[Y,[X,Y]]\\
&+\frac{1}{2}d_1d_2[[X,Y],X]-\frac{1}{2}d_3[X,Y]-\frac{1}{3}d_1d_3[X,[X,Y]]-\frac{1}{3}d_1d_3[Y,[X,Y]]\\
&+\frac{1}{2}d_1d_3[[X,Y],X]-\frac{1}{3}d_2d_3[X,[X,Y]]-\frac{1}{3}d_2d_3[Y,[X,Y]]+\frac{1}{2}d_2d_3[[X,Y],X]\\
&-\frac{1}{4}d_1d_2d_3[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[X,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[X,[[X,Y],X]]\\
&-\frac{1}{4}d_1d_2d_3[Y,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_3[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_2d_3[Y,[[X,Y],X]]\\
&-\frac{1}{4}d_1d_2d_3[[[X,Y],X],X]+\frac{1}{4}d_1d_2d_3[[[X,Y],Y],X]-\frac{1}{2}d_4[X,Y]-\frac{1}{3}d_1d_4[X,[X,Y]]\\
&-\frac{1}{3}d_1d_4[Y,[X,Y]]+\frac{1}{2}d_1d_4[[X,Y],X]-\frac{1}{3}d_2d_4[X,[X,Y]]-\frac{1}{3}d_2d_4[Y,[X,Y]]\\
&+\frac{1}{2}d_2d_4[[X,Y],X]-\frac{1}{4}d_1d_2d_4[X,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_4[X,[Y,[X,Y]]]\\
&+\frac{1}{2}d_1d_2d_4[X,[[X,Y],X]]-\frac{1}{4}d_1d_2d_4[Y,[X,[X,Y]]]-\frac{1}{4}d_1d_2d_4[Y,[Y,[X,Y]]]\\
&+\frac{1}{2}d_1d_2d_4[Y,[[X,Y],X]]-\frac{1}{4}d_1d_2d_4[[[X,Y],X],X]+\frac{1}{4}d_1d_2d_4[[[X,Y],Y],X]\\
&-\frac{1}{3}d_3d_4[X,[X,Y]]-\frac{1}{3}d_3d_4[Y,[X,Y]]+\frac{1}{2}d_3d_4[[X,Y],X]-\frac{1}{4}d_1d_3d_4[X,[X,[X,Y]]]\\
&-\frac{1}{4}d_1d_3d_4[X,[Y,[X,Y]]]+\frac{1}{2}d_1d_3d_4[X,[[X,Y],X]]-\frac{1}{4}d_1d_3d_4[Y,[X,[X,Y]]]\\
&-\frac{1}{4}d_1d_3d_4[Y,[Y,[X,Y]]]+\frac{1}{2}d_1d_3d_4[Y,[[X,Y],X]]-\frac{1}{4}d_1d_3d_4[[[X,Y],X],X]\\
&+\frac{1}{4}d_1d_3d_4[[[X,Y],Y],X]-\frac{1}{4}d_2d_3d_4[X,[X,[X,Y]]]-\frac{1}{4}d_2d_3d_4[X,[Y,[X,Y]]]\\
&+\frac{1}{2}d_2d_3d_4[X,[[X,Y],X]]-\frac{1}{4}d_2d_3d_4[Y,[X,[X,Y]]]-\frac{1}{4}d_2d_3d_4[Y,[Y,[X,Y]]]\\
&+\frac{1}{2}d_2d_3d_4[Y,[[X,Y],X]]-\frac{1}{4}d_2d_3d_4[[[X,Y],X],X]+\frac{1}{4}d_2d_3d_4[[[X,Y],Y],X]\\
&-\frac{1}{5}d_1d_2d_3d_4[X,[X,[X,[X,Y]]]]-\frac{1}{5}d_1d_2d_3d_4[X,[X,[Y,[X,Y]]]]\\
&+\frac{1}{2}d_1d_2d_3d_4[X,[X,[[X,Y],X]]]-\frac{1}{5}d_1d_2d_3d_4[X,[Y,[X,[X,Y]]]]\\
&-\frac{1}{5}d_1d_2d_3d_4[X,[Y,[Y,[X,Y]]]]+\frac{1}{2}d_1d_2d_3d_4[X,[Y,[[X,Y],X]]]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{3}d_1d_2d_3d_4\left[X,\left[\left[\left[X,Y\right],X\right],X\right]\right]+\frac{1}{3}d_1d_2d_3d_4\left[X,\left[\left[\left[X,Y\right],Y\right],X\right]\right]\\
&-\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]-\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],X\right]\right]\right]-\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&-\frac{1}{5}d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],X\right]\right]\right]\\
&-\frac{1}{3}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],X\right],X\right]\right]+\frac{1}{3}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],Y\right],X\right]\right]\\
&-\frac{1}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right]\\
&+d_1d_2d_3d_4\left[\left[X,Y\right],\left[\left[X,Y\right],X\right]\right]+\frac{1}{8}d_1d_2d_3d_4\left[\left[X,\left[\left[X,Y\right],X\right]\right],X\right]\\
&+\frac{1}{8}d_1d_2d_3d_4\left[\left[X,\left[\left[X,Y\right],Y\right]\right],X\right]+\frac{1}{8}d_1d_2d_3d_4\left[\left[Y,\left[\left[X,Y\right],X\right]\right],X\right]\\
&+\frac{1}{8}d_1d_2d_3d_4\left[\left[Y,\left[\left[X,Y\right],Y\right]\right],X\right]+\frac{1}{3}d_1d_2d_3d_4\left[\left[Y,\left[\left[X,Y\right],X\right]\right],\left[X,Y\right]\right]\\
&-\frac{1}{3}d_1d_2d_3d_4\left[\left[\left[X,Y\right],Y\right],\left[X,Y\right]\right]+\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],X\right],X\right],X\right]\\
&-\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],X\right],Y\right],X\right]-\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],Y\right],X\right],X\right]\\
&+\frac{1}{8}d_1d_2d_3d_4\left[\left[\left[\left[X,Y\right],Y\right],Y\right],X\right] \qquad (26)
\end{aligned}
$$

Letting m_{51} be the right-hand side of (26) with the first term X deleted, we have

(25)

$= \exp\ -d_5 m_{51}. \exp\ d_5 X + d_5 m_{51}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4)(X+Y) + d_5 Y + \frac{1}{2}(d_1+d_2+d_3+d_4)^2 [X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3 [[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4 ([X+Y,[[X,Y],X+Y]] + [[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 3(

$= \exp\ -d_5 m_{51}. (X + m_{51})_{d_5}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4)(X+Y) + d_5 Y + \frac{1}{2}(d_1+d_2+d_3+d_4)^2 [X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3 [[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4 ([X+Y,[[X,Y],X+Y]] + [[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 4(

$= \exp\ -d_5 m_{51}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y) + \frac{1}{2}(d_1+d_2+d_3+d_4)^2 [X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3 [[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4 ([X+Y,[[X,Y],X+Y]] + [[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51} + d_5 i_{52} + d_5 i_{53} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By (26)((27)

We let j_{51} be the result of $-m_{51}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, due to Theorem 2, we have

$$\delta^{\mathrm{right}}(\exp)\begin{pmatrix} (d_1+d_2+d_3+d_4+d_5)(X+Y)+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[X+Y,[[X,Y],X+Y]] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4[[[X,Y],X-Y],X-Y] \\ +d_5i_{51}+d_5i_{52}+d_5i_{53} \end{pmatrix}(j_{51})$$

$$\begin{aligned}
&=\frac{1}{2}d_1[X,Y]+\frac{1}{2}d_2[X,Y]+\frac{5}{6}d_1d_2[X,[X,Y]]+\frac{5}{6}d_1d_2[Y,[X,Y]]-\frac{1}{2}d_1d_2[[X,Y],X]\\
&+\frac{1}{2}d_3[X,Y]+\frac{5}{6}d_1d_3[X,[X,Y]]+\frac{5}{6}d_1d_3[Y,[X,Y]]-\frac{1}{2}d_1d_3[[X,Y],X]\\
&+\frac{5}{6}d_2d_3[X,[X,Y]]+\frac{5}{6}d_2d_3[Y,[X,Y]]-\frac{1}{2}d_2d_3[[X,Y],X]+\frac{5}{4}d_1d_2d_3[X,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_2d_3[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_3[X,[[X,Y],X]]+\frac{5}{4}d_1d_2d_3[Y,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_2d_3[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_3[Y,[[X,Y],X]]+\frac{1}{4}d_1d_2d_3[[[X,Y],X],X]\\
&-\frac{1}{4}d_1d_2d_3[[[X,Y],Y],X]+\frac{1}{2}d_4[X,Y]+\frac{5}{6}d_1d_4[X,[X,Y]]+\frac{5}{6}d_1d_4[Y,[X,Y]]\\
&-\frac{1}{2}d_1d_4[[X,Y],X]+\frac{5}{6}d_2d_4[X,[X,Y]]+\frac{5}{6}d_2d_4[Y,[X,Y]]-\frac{1}{2}d_2d_4[[X,Y],X]\\
&+\frac{5}{4}d_1d_2d_4[X,[X,[X,Y]]]+\frac{5}{4}d_1d_2d_4[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_4[X,[[X,Y],X]]\\
&+\frac{5}{4}d_1d_2d_4[Y,[X,[X,Y]]]+\frac{5}{4}d_1d_2d_4[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_2d_4[Y,[[X,Y],X]]\\
&+\frac{1}{4}d_1d_2d_4[[[X,Y],X],X]-\frac{1}{4}d_1d_2d_4[[[X,Y],Y],X]+\frac{5}{6}d_3d_4[X,[X,Y]]\\
&+\frac{5}{6}d_3d_4[Y,[X,Y]]-\frac{1}{2}d_3d_4[[X,Y],X]+\frac{5}{4}d_1d_3d_4[X,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_3d_4[X,[Y,[X,Y]]]-\frac{5}{4}d_1d_3d_4[X,[[X,Y],X]]+\frac{5}{4}d_1d_3d_4[Y,[X,[X,Y]]]\\
&+\frac{5}{4}d_1d_3d_4[Y,[Y,[X,Y]]]-\frac{5}{4}d_1d_3d_4[Y,[[X,Y],X]]+\frac{1}{4}d_1d_3d_4[[[X,Y],X],X]\\
&-\frac{1}{4}d_1d_3d_4[[[X,Y],Y],X]+\frac{5}{4}d_1d_3d_4[X,[X,[X,Y]]]+\frac{5}{4}d_1d_3d_4[X,[Y,[X,Y]]]\\
&-\frac{5}{4}d_1d_3d_4[X,[[X,Y],X]]+\frac{5}{4}d_2d_3d_4[Y,[X,[X,Y]]]+\frac{5}{4}d_2d_3d_4[Y,[Y,[X,Y]]]\\
&-\frac{5}{4}d_2d_3d_4[Y,[[X,Y],X]]+\frac{1}{4}d_2d_3d_4[[[X,Y],X],X]-\frac{1}{4}d_2d_3d_4[[[X,Y],Y],X]\\
&+\frac{5}{3}d_1d_2d_3d_4[X,[X,[X,[X,Y]]]]+\frac{5}{3}d_1d_2d_3d_4[X,[X,[Y,[X,Y]]]]\\
&-2d_1d_2d_3d_4[X,[X,[[X,Y],X]]]+\frac{5}{3}d_1d_2d_3d_4[X,[Y,[X,[X,Y]]]]\\
&+\frac{5}{3}d_1d_2d_3d_4[X,[Y,[Y,[X,Y]]]]-2d_1d_2d_3d_4[X,[Y,[[X,Y],X]]]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[\left[\left[X,Y\right],X\right],X\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[X,\left[\left[\left[X,Y\right],Y\right],X\right]\right]\\
&+\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]+\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&-2d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],X\right]\right]\right]+\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{5}{3}d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]-2d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],X\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],X\right],X\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[\left[\left[X,Y\right],Y\right],X\right]\right]\\
&+2d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]+2d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right]\\
&-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[\left[X,Y\right],X\right]\right]-\frac{1}{2}d_1d_2d_3d_4\left[\left[\left[X,Y\right],X\right],\left[X,Y\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[\left[\left[X,Y\right],Y\right],\left[X,Y\right]\right] \qquad (28)
\end{aligned}
$$

Letting m_{52} be the right-hand side of (28), we have

$$
\begin{aligned}
&(27)\\
&=\exp\ -d_5m_{51}-d_5m_{52}.\exp\ d_5m_{52}\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4+d_5)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y]\\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y],Y-X]\\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}+d_5i_{53}\end{array}\right).\\
&\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}\\
&)\text{By Proposition 3}(\\
&=\exp\ -d_5m_{51}-d_5m_{52}.\,(m_{52})_{d_5}\,.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4+d_5)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y]\\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y],Y-X]\\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}+d_5i_{53}\end{array}\right).\\
&\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}\\
&)\text{By Proposition 4}(\\
&=\exp\ -d_5m_{51}-d_5m_{52}.\\
&\exp\left(\begin{array}{c}(d_1+d_2+d_3+d_4+d_5)\,(X+Y)+\frac{1}{2}\,(d_1+d_2+d_3+d_4)^2\,[X,Y]\\ +\frac{1}{12}\,(d_1+d_2+d_3+d_4)^3\,[[X,Y],Y-X]\\ +\frac{1}{96}\,(d_1+d_2+d_3+d_4)^4\,([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y])\\ +d_5i_{51}+d_5i_{52}+d_5i_{53}+d_5j_{51}\end{array}\right).\\
&\exp\ -d_5n_{51}-d_5n_{52}-d_5n_{53}-d_5n_{54}\\
&)\text{By (28)}(\qquad (29)
\end{aligned}
$$

We let j_{52} be the result of $-m_{51}-m_{52}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, by dint of Theorem 2, we have

$$\delta^{\mathrm{right}}\left(\exp\right)\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4+d_5\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right]\\+\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right]\\+\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right]\\+\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]\\+d_5i_{51}+d_5i_{52}+d_5i_{53}+d_5j_{51}\end{array}\right)\left(j_{52}\right)$$

$$\begin{aligned}
&=-\frac{1}{2}d_1d_2\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_1d_2\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_1d_3\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_1d_3\left[Y,\left[X,Y\right]\right]\\
&-\frac{1}{2}d_2d_3\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_2d_3\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_2d_3\left[X,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_3\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[X,\left[\left[X,Y\right],X\right]\right]-\frac{7}{4}d_1d_2d_3\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_3\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[Y,\left[\left[X,Y\right],X\right]\right]-\frac{1}{2}d_1d_4\left[X,\left[X,Y\right]\right]\\
&-\frac{1}{2}d_1d_4\left[Y,\left[X,Y\right]\right]-\frac{1}{2}d_2d_4\left[X,\left[X,Y\right]\right]-\frac{1}{2}d_2d_4\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_2d_4\left[X,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[X,\left[\left[X,Y\right],X\right]\right]-\frac{7}{4}d_1d_2d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{7}{4}d_1d_2d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[Y,\left[\left[X,Y\right],X\right]\right]-\frac{1}{2}d_3d_4\left[X,\left[X,Y\right]\right]\\
&-\frac{1}{2}d_3d_4\left[Y,\left[X,Y\right]\right]-\frac{7}{4}d_1d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_1d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_3d_4\left[X,\left[\left[X,Y\right],X\right]\right]-\frac{7}{4}d_1d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_1d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_3d_4\left[Y,\left[\left[X,Y\right],X\right]\right]-\frac{7}{4}d_2d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_2d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[X,\left[\left[X,Y\right],X\right]\right]-\frac{7}{4}d_2d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]-\frac{7}{4}d_2d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[Y,\left[\left[X,Y\right],X\right]\right]-3d_1d_2d_3d_4\left[X,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&-3d_1d_2d_3d_4\left[X,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[X,\left[\left[X,Y\right],X\right]\right]\right]\\
&-3d_1d_2d_3d_4\left[X,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]-3d_1d_2d_3d_4\left[X,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[Y,\left[\left[X,Y\right],X\right]\right]\right]-3d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&-3d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]-3d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&-3d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[\left[X,Y\right],X\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[\left[X,Y\right],X\right]\right]\right]-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[X,\left[X,Y\right]\right]\right]\\
&-\frac{3}{2}d_1d_2d_3d_4\left[\left[X,Y\right],\left[Y,\left[X,Y\right]\right]\right]
\end{aligned} \tag{30}$$

Letting m_{53} be the right-hand side of (30), we have

(29)

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}. \exp\ d_5 m_{53}$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y)+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 3(

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}. (m_{53})_{d_5}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y)+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 4(

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y)+\frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51}+d_5 j_{52} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By (30)((31)

We let j_{53} be the result of $-m_{51}-m_{52}-m_{53}$ by deleting all the terms whose coefficients contain $d_1d_2d_3d_4$. Then, thanks to Theorem 2, we have

$$\delta^{\mathrm{right}}\left(\exp\right)\left(\begin{array}{c}\left(d_1+d_2+d_3+d_4+d_5\right)\left(X+Y\right)+\frac{1}{2}\left(d_1+d_2+d_3+d_4\right)^2\left[X,Y\right]\\ +\frac{1}{12}\left(d_1+d_2+d_3+d_4\right)^3\left[\left[X,Y\right],Y-X\right]\\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[X+Y,\left[\left[X,Y\right],X+Y\right]\right]\\ +\frac{1}{96}\left(d_1+d_2+d_3+d_4\right)^4\left[\left[\left[X,Y\right],X-Y\right],X-Y\right]\\ +d_5i_{51}+d_5i_{52}+d_5i_{53}+d_5j_{51}+d_5j_{52}\end{array}\right)\left(j_{53}\right)$$

$$\begin{aligned}
&=\frac{3}{4}d_1d_2d_3\left[X,\left[X,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_3\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_2d_3\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[X,\left[X,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_2d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_2d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_1d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]+\frac{3}{4}d_1d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[X,\left[X,\left[X,Y\right]\right]\right]+\frac{3}{4}d_2d_3d_4\left[X,\left[Y,\left[X,Y\right]\right]\right]+\frac{3}{4}d_2d_3d_4\left[Y,\left[X,\left[X,Y\right]\right]\right]\\
&+\frac{3}{4}d_2d_3d_4\left[Y,\left[Y,\left[X,Y\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[X,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[X,\left[Y,\left[X,Y\right]\right]\right]\right]+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[X,\left[X,Y\right]\right]\right]\right]\\
&+\frac{1}{2}d_1d_2d_3d_4\left[Y,\left[Y,\left[Y,\left[X,Y\right]\right]\right]\right]
\end{aligned}\tag{32}$$

Letting m_{54} be the right-hand side of (32), we have

(31)

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}. \exp\ d_5 m_{54}$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y) + \frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51}+d_5 j_{52} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 3(

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}. (m_{54})_{d_5}$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y) + \frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51}+d_5 j_{52} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By Proposition 4(

$= \exp\ -d_5 m_{51} - d_5 m_{52} - d_5 m_{53} - d_5 m_{54}.$

$$\exp\left(\begin{array}{c} (d_1+d_2+d_3+d_4+d_5)(X+Y) + \frac{1}{2}(d_1+d_2+d_3+d_4)^2[X,Y] \\ +\frac{1}{12}(d_1+d_2+d_3+d_4)^3[[X,Y],Y-X] \\ +\frac{1}{96}(d_1+d_2+d_3+d_4)^4([X+Y,[[X,Y],X+Y]]+[[[X,Y],X-Y],X-Y]) \\ +d_5 i_{51}+d_5 i_{52}+d_5 i_{53}+d_5 j_{51}+d_5 j_{52}+d_5 j_{53} \end{array}\right).$$

$\exp\ -d_5 n_{51} - d_5 n_{52} - d_5 n_{53} - d_5 n_{54}$

)By (32)((33)

Since the coefficient of every term in $-m_{51}-m_{52}-m_{53}-m_{54}$ contains $d_1d_2d_3d_4$, we are done. We have

$$
\begin{aligned}
& i_{51}+i_{52}+i_{53}+j_{51}+j_{52}+j_{53} \\
& =d_1\,[X,Y]+d_2\,[X,Y]-\frac{1}{2}d_1d_2\,[[X,Y]\,,X]+\frac{1}{2}d_1d_2\,[[X,Y]\,,Y]+d_3\,[X,Y] \\
& -\frac{1}{2}d_1d_3\,[[X,Y]\,,X]+\frac{1}{2}d_1d_3\,[[X,Y]\,,Y]-\frac{1}{2}d_2d_3\,[[X,Y]\,,X] \\
& +\frac{1}{2}d_2d_3\,[[X,Y]\,,Y]+\frac{1}{4}d_1d_2d_3\,[X,[[X,Y]\,,X]]+\frac{1}{4}d_1d_2d_3\,[X,[[X,Y]\,,Y]] \\
& +\frac{1}{4}d_1d_2d_3\,[Y,[[X,Y]\,,X]]+\frac{1}{4}d_1d_2d_3\,[Y,[[X,Y]\,,Y]]+\frac{1}{4}d_1d_2d_3\,[[[X,Y]\,,X]\,,X] \\
& -\frac{1}{4}d_1d_2d_3\,[[[X,Y]\,,X]\,,Y]-\frac{1}{4}d_1d_2d_3\,[[[X,Y]\,,Y]\,,X]+\frac{1}{4}d_1d_2d_3\,[[[X,Y]\,,Y]\,,Y] \\
& +d_4\,[X,Y]-\frac{1}{2}d_1d_4\,[[X,Y]\,,X]+\frac{1}{2}d_1d_4\,[[X,Y]\,,Y]-\frac{1}{2}d_2d_4\,[[X,Y]\,,X] \\
& +\frac{1}{2}d_2d_4\,[[X,Y]\,,Y]+\frac{1}{4}d_1d_2d_4\,[X,[[X,Y]\,,X]]+\frac{1}{4}d_1d_2d_4\,[X,[[X,Y]\,,Y]] \\
& +\frac{1}{4}d_1d_2d_4\,[Y,[[X,Y]\,,X]]+\frac{1}{4}d_1d_2d_4\,[Y,[[X,Y]\,,Y]]+\frac{1}{4}d_1d_2d_4\,[[[X,Y]\,,X]\,,X] \\
& -\frac{1}{4}d_1d_2d_4\,[[[X,Y]\,,X]\,,Y]-\frac{1}{4}d_1d_2d_4\,[[[X,Y]\,,Y]\,,X]+\frac{1}{4}d_1d_2d_4\,[[[X,Y]\,,Y]\,,Y] \\
& -\frac{1}{2}d_3d_4\,[[X,Y]\,,X]+\frac{1}{2}d_3d_4\,[[X,Y]\,,Y]+\frac{1}{4}d_1d_3d_4\,[X,[[X,Y]\,,X]] \\
& +\frac{1}{4}d_1d_3d_4\,[X,[[X,Y]\,,Y]]+\frac{1}{4}d_1d_3d_4\,[Y,[[X,Y]\,,X]]+\frac{1}{4}d_1d_3d_4\,[Y,[[X,Y]\,,Y]] \\
& +\frac{1}{4}d_1d_3d_4\,[[[X,Y]\,,X]\,,X]-\frac{1}{4}d_1d_3d_4\,[[[X,Y]\,,X]\,,Y]-\frac{1}{4}d_1d_3d_4\,[[[X,Y]\,,Y]\,,X] \\
& +\frac{1}{4}d_1d_3d_4\,[[[X,Y]\,,Y]\,,Y]+\frac{1}{4}d_2d_3d_4\,[X,[[X,Y]\,,X]]+\frac{1}{4}d_2d_3d_4\,[X,[[X,Y]\,,Y]] \\
& +\frac{1}{4}d_2d_3d_4\,[Y,[[X,Y]\,,X]]+\frac{1}{4}d_2d_3d_4\,[Y,[[X,Y]\,,Y]]+\frac{1}{4}d_2d_3d_4\,[[[X,Y]\,,X]\,,X] \\
& -\frac{1}{4}d_2d_3d_4\,[[[X,Y]\,,X]\,,Y]-\frac{1}{4}d_2d_3d_4\,[[[X,Y]\,,Y]\,,X]+\frac{1}{4}d_2d_3d_4\,[[[X,Y]\,,Y]\,,Y]
\end{aligned}
$$

on the one hand, and

$$
\begin{aligned}
& -m_{51}-m_{52}-m_{53}-m_{54}-n_{51}-n_{52}-n_{53}-n_{54} \\
& = d_1 d_2 d_3 d_4 \left[X,\left[X,\left[\left[X,Y\right],X\right]\right]\right] - d_1 d_2 d_3 d_4 \left[X,\left[X,\left[\left[X,Y\right],Y\right]\right]\right] \\
& + d_1 d_2 d_3 d_4 \left[X,\left[Y,\left[\left[X,Y\right],X\right]\right]\right] - d_1 d_2 d_3 d_4 \left[X,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right] \\
& + d_1 d_2 d_3 d_4 \left[Y,\left[X,\left[\left[X,Y\right],X\right]\right]\right] - d_1 d_2 d_3 d_4 \left[Y,\left[X,\left[\left[X,Y\right],Y\right]\right]\right] \\
& + d_1 d_2 d_3 d_4 \left[Y,\left[Y,\left[\left[X,Y\right],X\right]\right]\right] - d_1 d_2 d_3 d_4 \left[Y,\left[Y,\left[\left[X,Y\right],Y\right]\right]\right] \\
& - \frac{1}{6} d_1 d_2 d_3 d_4 \left[X,\left[\left[\left[X,Y\right],X\right],X\right]\right] - \frac{1}{6} d_1 d_2 d_3 d_4 \left[X,\left[\left[\left[X,Y\right],X\right],Y\right]\right] \\
& + \frac{1}{6} d_1 d_2 d_3 d_4 \left[X,\left[\left[\left[X,Y\right],Y\right],X\right]\right] + \frac{1}{6} d_1 d_2 d_3 d_4 \left[X,\left[\left[\left[X,Y\right],Y\right],Y\right]\right] \\
& - \frac{1}{6} d_1 d_2 d_3 d_4 \left[Y,\left[\left[\left[X,Y\right],X\right],X\right]\right] - \frac{1}{6} d_1 d_2 d_3 d_4 \left[Y,\left[\left[\left[X,Y\right],X\right],Y\right]\right] \\
& + \frac{1}{6} d_1 d_2 d_3 d_4 \left[Y,\left[\left[\left[X,Y\right],Y\right],X\right]\right] + \frac{1}{6} d_1 d_2 d_3 d_4 \left[Y,\left[\left[\left[X,Y\right],Y\right],Y\right]\right] \\
& + \frac{1}{2} d_1 d_2 d_3 d_4 \left[\left[X,Y\right],\left[\left[X,Y\right],X\right]\right] + \frac{1}{2} d_1 d_2 d_3 d_4 \left[\left[X,Y\right],\left[\left[X,Y\right],Y\right]\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[X,\left[\left[X,Y\right],X\right]\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[X,\left[\left[X,Y\right],X\right]\right],Y\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[X,\left[\left[X,Y\right],Y\right]\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[X,\left[\left[X,Y\right],Y\right]\right],Y\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[Y,\left[\left[X,Y\right],X\right]\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[Y,\left[\left[X,Y\right],X\right]\right],Y\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[Y,\left[\left[X,Y\right],Y\right]\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[Y,\left[\left[X,Y\right],Y\right]\right],Y\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],X\right],X\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],X\right],X\right],Y\right] \\
& + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],X\right],Y\right],X\right] - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],X\right],Y\right],Y\right] \\
& + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],Y\right],X\right],X\right] - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],Y\right],X\right],Y\right] \\
& - \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],Y\right],Y\right],X\right] + \frac{1}{8} d_1 d_2 d_3 d_4 \left[\left[\left[\left[X,Y\right],Y\right],Y\right],Y\right]
\end{aligned}
$$

on the other. Therefore we have the desired result. ■

References

[1] Kock, Anders:Synthetic Differential Geometry, 2nd ed., London Mathematical Society Lecture Note Series **333**, Cambridge University Press, Cambridge, 2006.

[2] Kriegl, Andreas and Michor, Peter W.: The Convenient Setting of Global Analysis, American Mathematical Society, Rhode Island, 1997.

[3] Lavendhomme, René:Basic Concepts of Synthetic Differential Geometry, Kluwer Academic Publishers, Dordrecht, 1996.

[4] Nishimura, Hirokazu: The Baker-Campbell-Hausdorff formula and the Zassenhaus formula in synthetic differential geometry, Mathematics for Applications, **2** (2013), 61-91.

[5] Takamiya, Hirowaki:Mechanical derivation of the infinitesimal Baker-Campbell-Hausdorff formula, Master thesis, University of Tsukuba, March 2015 (Japanese).